



# THE HESS CASE IN RIGID-BODY DYNAMICS†

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Generalizations of the Hess integral are presented for different forms of the equations of motions of rigid body. The general conditions for the existence of this integral, which is due to the presence of additional explicit symmetries of the equations of motion, are pointed out. Problems of reducing the order, of the explicit integration and the qualitative analysis of the motion of a rigid body subject to these conditions are considered. Analogues of Hess cases for a gyroscope in gimbals and the Chaplygin equations describing the fall of a rigid body in a fluid are indicated for the first time. © 2003 Elsevier Science Ltd. All rights reserved.

## 1. THE HESS CASE IN THE EULER–POISSON EQUATIONS

In Hamilton form, the Euler–Poisson equations describing the motion of a heavy rigid body with a fixed point have the form

$$\dot{\mathbf{M}} = \mathbf{M} \times \frac{\partial H}{\partial \mathbf{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \mathbf{M}} \quad (1.1)$$

The Hamiltonian can be represented in the form

$$H = \frac{1}{2}(\mathbf{M}, \mathbf{A}\mathbf{M}) - \mu(\mathbf{r}, \boldsymbol{\gamma})$$

where  $\mathbf{M}$  is the kinetic momentum vector in the system of coordinates associated with the body,  $\boldsymbol{\gamma}$  is the unit vector of the vertical in the same system,  $\mathbf{A} = \mathbf{\Gamma}^{-1}$  is the inverse inertia tensor and  $\mathbf{r}$  is the radius vector of the centre of mass of the body in the fixed system.

The Poisson brackets for the variables  $\mathbf{M}$  and  $\boldsymbol{\gamma}$  are as follows:

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k \quad \{\gamma_i, \gamma_j\} = 0 \quad (1.2)$$

In the case of any Hamilton function, Eqs (1.1) allow of an area integral and a geometric integral of the form

$$F_1 = (\mathbf{M}, \boldsymbol{\gamma}), \quad F_2 = \boldsymbol{\gamma}^2 = 1 \quad (1.3)$$

Only several general and special cases of the integrability of Eqs (1.1) are known for which, apart from these integrals, a further, additional, general (particular) integral exists. These cases are realized with additional constraints on the parameters of the system and on the initial conditions. They are the Euler, Lagrange, Kovalevskaya and Goryachev–Chaplygin cases. In the general case, Eqs (1.1) are unintegrable [1].

Apart from the integrable cases of Eqs (1.1), some particular solutions are known, the majority of which have been cited in [2], for example. The most well known is the Hess solution which is defined by an invariant relation which is linear in the momentum  $\mathbf{M}$ .

For the Hess case, the Hamiltonian of the Euler–Poisson equations has the form [3]

$$H = \frac{1}{2}(a_1M_1^2 + a_2M_2^2 + a_3M_3^2) - \mu(\sqrt{a_2 - a_1}\gamma_1 \pm \sqrt{a_3 - a_2}\gamma_3) \quad (1.4)$$

$$a_1 < a_2 < a_3$$

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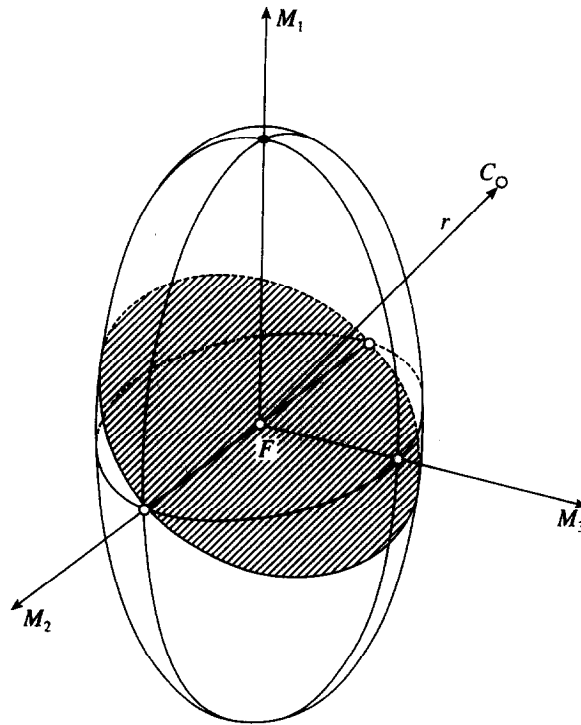


Fig. 1

that is, the centre of mass in this case lies on the axis which is perpendicular to the circular cross-section of a gyration ellipsoid (Fig. 1).

The invariant relation pointed out by Hess has the form

$$F = \sqrt{a_2 - a_1} M_1 \pm \sqrt{a_3 - a_2} M_3 = 0 \quad (1.5)$$

The different signs correspond to different circular cross-sections.

An analytical investigation of Hess motion was carried out by Nekrasov [4] and, instead of elliptic quadratures for a single variable, the Riccati equation is obtained. It is easily obtained (in the case of a variable  $l$ ) if Andoyer–Deprit variables are used [5]. A geometrical interpretation as a “loxodromic pendulum” has been given by Zhukovskii [6]. Later, using the Kovalevskaya method, an invariant Hess relation was given by Appel’rot [7] which attempted to fill certain gaps in Kovalevskaya’s work regarding the uniqueness of the general solution of the Euler–Poisson equations.

*Remarks.* 1. The solution in the Hess case branches in the complex time plane.

2. The dynamics of the reduced system can be different depending on the cyclic variable used in the reduction (the angle of natural rotation or the precession angle) [5].

The dynamics of the reduced system (1.4) in the variables  $(M, \gamma)$  at high energies have been described previously [1]. In these variables, the integral (1.5) determines the specific torus on which unstable periodic solutions are found which, when  $\mu \rightarrow 0$ , correspond to permanent rotations about the central axis. In this case, the torus is filled with trajectories which asymptotically approximate to these solutions. In a perturbation of the Euler–Poisson problem, for which the Hess conditions are satisfied, it is found that the pair of separatrices, corresponding to unstable permanent rotations, do not split.

The description of the dynamics of the reduced system (with asymptotic behaviour) does not contradict Zhukovskii’s result, according to which the centre of mass of a body executes a quasi-periodic motion according to the law for a spherical pendulum since, unlike Eqs (1.1), the system describing the centre of mass is obtained by a reduction with respect to the angle of natural rotation about an axis perpendicular to a circular cross-section [5], rather than with respect to the angle of precession.

## 2. ANALOGY WITH THE LAGRANGE CASE, CYCLIC VARIABLE

The Hess case is similar in many respects to the Lagrange case and is associated with the existence in the system of a cyclic variable (an explicit symmetry of the Hamiltonian with respect to rotations) in

one of the levels of a certain “cyclic” integral. In order to show this explicitly, we will write the Hamiltonian (1.4) in a system of coordinates for which one of the axes, the  $Ox_3$  axis (Fig. 1) coincides with the axis which is perpendicular to the circular cross-section of the gyration ellipsoid (compare with another approach in [8])

$$H = \frac{1}{2}(a'_1(M_1^2 + M_2^2) + a'_3M_3^2 + 2bM_3M_1) - \mu'\gamma_3 \quad (2.1)$$

Terms of the form  $M_3M_2$  can be eliminated from the Hamiltonian by a rotation of the  $Ox_1, Ox_2$  axes. (The transition matrix is determined by formula (3.8).) The Hess integral (1.5) now takes the form

$$M_3 = 0 \quad (2.2)$$

The Hamiltonian (2.1) at the level  $M_3 = 0$  is identical with the Lagrange Hamiltonian [5] and in order to discuss the reduced system which describes the dynamics of the angle of nutation of the centre of mass  $\gamma_3 = \cos\theta$ , we can use the variables (for greater detail see [9])

$$\begin{aligned} \mathbf{K} &= (K_1, K_2), \quad K_1 = (M_1\gamma_1 + M_2\gamma_2)/\tilde{\gamma}, \quad K_2 = (M_1\gamma_2 - M_2\gamma_1)/\tilde{\gamma} \\ \boldsymbol{\sigma} &= (\sigma_1, \sigma_2), \quad \sigma_1 = \tilde{\gamma}, \quad \sigma_2 = \gamma_3, \quad \tilde{\gamma} = \sqrt{\gamma_1^2 + \gamma_2^2} \end{aligned} \quad (2.3)$$

At the level  $M_3 = 0$ , they form a closed system of equations

$$\begin{aligned} \dot{K}_1 &= -a'_1K_1K_2\sigma_2/\sigma_1, \quad \dot{K}_2 = a'_1K_1^2\sigma_2/\sigma_1 - \mu'\sigma_1 \\ \dot{\sigma}_1 &= a'_1K_2\sigma_2, \quad \dot{\sigma}_2 = -a'_1K_2\sigma_1 \end{aligned} \quad (2.4)$$

The Hamiltonian (2.1) can be written in the form

$$H = \frac{1}{2}\mathbf{K}^2 - \mu\sigma_2 + \frac{1}{2}M_3(a'_3M_3 + 2bM_1)$$

The equation for  $\sigma_2$  is identical with the quadrature for the vertical coordinate of a spherical pendulum [6]

$$\dot{\sigma}_2 = 2mR^2a_2(1 - \sigma_2^2)\left(h - \mu\sigma - \frac{1}{2}c^2/(1 - \sigma_2^2)\right)$$

where  $R$  is the distance from the centre of mass to the fixing point,  $a_2^{-1} = I_2$  is the average principal moment of inertia,  $H = h$  and  $(\mathbf{M}, \boldsymbol{\gamma}) = c$  are constants of the integrals.

The precession angle  $\psi$  in this case (as in the Lagrange case) is completely defined by the solution of the reduced system (2.4).

$$\dot{\psi} = a'_1K_1/\sigma_1$$

and is independent of the solution for the angle of natural rotation  $\varphi(t)$ .

*Remark.* Levi-Civita has made a detailed study of the reduction in the order when there are invariant relations which are linear in the momenta. His basic results are contained in the well-known textbook [10]. However, in applying his results to the dynamics of a rigid body, he paid no attention to the Hess case, concentrating on a more particular class of invariant relations which determine Staude rotations.

### 3. CONDITIONS FOR THE EXISTENCE OF A HESS INTEGRAL FOR A GENERALIZED POTENTIAL FIELD

We will now consider the extension of the Hess integral to the case when the potential depends on three fields and a generalized potential is present in the Hamiltonian

$$H = \frac{1}{2}(\mathbf{M}, \mathbf{A}'\mathbf{M}) + (\mathbf{M}, \mathbf{W}(\alpha, \beta, \gamma)) + U(\alpha, \beta, \gamma) \quad (3.1)$$

where  $\mathbf{A}' = \|a'_{ij}\|$  is a constant but not necessarily diagonal matrix and  $\alpha, \beta, \gamma$  are the projections of the unit vectors of the fixed system of coordinates onto the axes associated with the body.

A particular integral of the form

$$M_3 - c = 0, \quad c = \text{const} \quad (3.2)$$

is a natural generalization of the Hess integral (1.5) in the special system of coordinates (for which the centre of mass lies on the  $O_z$  axis).

The conditions for its existence can be represented in the general form

$$\begin{aligned} a'_{11} &= a'_{22}, \quad a'_{12} = 0 \\ \hat{L}(U + cW_3) &= 0, \quad \hat{L}W_1 + W_2 + ca'_{23} = 0, \quad \hat{L}W_2 - W_1 - ca'_{13} = 0 \\ \hat{L} &= (\hat{L}_\alpha + \hat{L}_\beta + \hat{L}_\gamma) \\ \hat{L}_\delta &= \delta_1 \frac{\partial}{\partial \delta_2} - \delta_2 \frac{\partial}{\partial \delta_1}, \quad \delta = \alpha, \beta, \gamma \end{aligned} \quad (3.3)$$

We also give, in explicit form, the conditions for the existence of the Hess integral (3.2) for the particular form of system (3.1) for which

$$\mathbf{W} = \mathbf{K} + \sum_{i=1}^3 \mathbf{B}^{(i)} \mathbf{e}_i, \quad U = \sum_{i=1}^3 (\mathbf{r}_i, \mathbf{e}_i) + \frac{1}{2} \sum_{i=1}^3 (\mathbf{e}_i, \mathbf{C}^{(i)} \mathbf{e}_i) \quad (3.4)$$

where  $\mathbf{K}$  and  $\mathbf{r}_i$  are constant vectors,  $\mathbf{e}_1 = \alpha$ ,  $\mathbf{e}_2 = \beta$ ,  $\mathbf{e}_3 = \gamma$  and  $\mathbf{C}^{(i)}$  are symmetric and  $\mathbf{B}^{(i)}$  are arbitrary  $3 \times 3$  matrices ( $i = 1, 2, 3$ ).

The conditions for the existence of a Hess integral for certain cases of system (3.1) with the potential (3.4) have been given in [11, 12].

Using relations (3.2) and (3.3), we find

$$\begin{aligned} b_{11}^{(i)} &= b_{22}^{(i)}, \quad b_{12}^{(i)} = -b_{12}^{(i)}, \quad b_{13}^{(i)} = b_{23}^{(i)} = 0 \\ \mathbf{C}^{(i)} &= \text{diag}(c_{11}^{(i)}, c_{11}^{(i)}, c_{33}^{(i)}), \quad \mathbf{K} = (-ca'_{13}, -ca'_{23}, k_3 a'_{33}), \quad \mathbf{r}_i = (cb_{31}^{(i)}, cb_{32}^{(i)}, r_3^i) \end{aligned}$$

where  $k_3$  and  $r_3^i$  ( $i = 1, 2, 3$ ) are arbitrary constants. The Hamiltonian can be represented in the explicit form

$$\begin{aligned} H &= \frac{1}{2}(a'_{11}(M_1^2 + M_2^2) + a'_{33}(M_3 + k_3)^2) + (M_3 - c)(a'_{13}M_1 + a'_{23}M_2) + \\ &+ b_{11}^{(1)}(M_1\alpha_1 + M_2\alpha_2) + b_{12}^{(1)}(M_1\alpha_2 - M_2\alpha_1)b_{33}^{(1)}M_3\alpha_3 + \\ &+ (M_3 - c)(b_{31}^{(1)}\alpha_1 + b_{32}^{(1)}\alpha_2) + \frac{1}{2}(c_{11}^{(1)}(\alpha_1^2 + \alpha_2^2) + c_{33}^{(1)}\alpha_3^2) + r_3^{(1)}\alpha_3 + \dots \end{aligned} \quad (3.5)$$

where the analogous terms containing  $\beta$  and  $\gamma$  have been omitted.

Using the kinetic momentum vector in the fixed axes  $\mathbf{N} = ((\mathbf{M}, \alpha), (\mathbf{M}, \beta), (\mathbf{M}, \gamma))$  and the vector  $\mathbf{p} = (\alpha_3, \beta_3, \gamma_3)$  [5], the Hamiltonian (3.5) can be represented in the explicit form

$$\begin{aligned} H &= \frac{1}{2}a'_{11}\mathbf{N}^2 + (\mathbf{b}_1, \mathbf{N}) + (\mathbf{b}_2 \times \mathbf{p}, \mathbf{N}) + (\mathbf{r} + c\mathbf{b}_3 - c\mathbf{b}_1, \mathbf{p}) + \\ &+ \frac{1}{2}(\mathbf{p}, \mathbf{C}\mathbf{p}) + (M_3 - c)f(\mathbf{M}, \alpha, \beta, \gamma) \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} \mathbf{b}_1 &= (b_{11}^{(1)}, b_{11}^{(2)}, b_{11}^{(3)}), \quad \mathbf{b}_2 = (b_{12}^{(1)}, b_{12}^{(2)}, b_{12}^{(3)}), \quad \mathbf{b}_3 = (b_{33}^{(1)}, b_{33}^{(2)}, b_{33}^{(3)}) \\ \mathbf{C} &= \text{diag}(c_{33}^{(1)} - c_{11}^{(1)}, c_{33}^{(2)} - c_{11}^{(2)}, c_{33}^{(3)} - c_{11}^{(3)}), \quad \mathbf{r} = (r_3^{(1)}, r_3^{(2)}, r_3^{(3)}) \end{aligned}$$

in which case the function  $f(\mathbf{M}, \alpha, \beta, \gamma)$  cannot be expressed in terms of the variables  $\mathbf{N}$  and  $\mathbf{p}$  since, otherwise, a reduction to an arbitrary level of the integral  $F = M_3$ , which corresponds to the Lagrange case, would be possible.

The Poisson bracket for the variables  $N_i, p_i, (i = 1, 2, 3)$  has the form

$$\{N_i, N_j\} = \epsilon_{ijk}N_k, \quad \{N_i, p_j\} = \epsilon_{ijk}p_k, \quad \{p_i, p_j\} = 0 \quad (3.7)$$

Since the vectors  $\mathbf{N}$  and  $\mathbf{p}$  commute with the quantity  $M_3 = (\mathbf{N}, \mathbf{p})$ , the equations of motion for them at the level  $M_3 = c$  separate and are described by a Hamiltonian system in an  $e(3)$  algebra with the Hamiltonian (3.6) taken with the condition  $M_3 - c = 0$ , that is, they are described by a system with two degrees of freedom.

Hence, we obtain the following result: the phase flow of system (3.5) subject to the Hess condition is isomorphous with the flow of a spherical top (3.6) at a fixed level of constant area  $(\mathbf{N}, \mathbf{p}) = M_3 = c$ . Consequently, the conditions of integrability at the level of the Hess integral are determined by the cases of integrability of a spherical top.

Thus, when  $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b}_3 = \mathbf{r} = 0$ , we obtain the integrable system of the Clebsch case (which, when  $c = 0$ , is also identical to the Neumann system) and, when  $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b}_3 = 0$  and  $\mathbf{C} = 0$ , we obtain the Lagrange case for a single field.

*Remark.* The general system (3.6) has three degrees of freedom and, hence, a further additional integral is necessary for it to be integrable at the level of the Hess integral. In the general case, it does not exist.

A special system of coordinates has been used above, the axes of which do not coincide with the principal axes of the body and, in this system, the matrix  $\mathbf{A}$  is non-diagonal. The matrix  $\mathbf{A}$  can be transformed to diagonal form using the matrix

$$\begin{aligned} \mathbf{U} = \|u_{ij}\|; \quad u_{11} = u_{33} &= \sqrt{\frac{a_3 - a_2}{a_3 - a_1}}, \quad u_{22} = 1 \\ u_{13} = -u_{31} &= -\sqrt{\frac{a_2 - a_1}{a_3 - a_1}}, \quad u_{12} = u_{21} = u_{23} = u_{32} = 0 \end{aligned} \quad (3.8)$$

The conditions, imposed on the constants in the Hamiltonian (3.6) in this system of coordinates for the case of a single field, have been indicated by Hess [3].

In a system of coordinates for which the inertial tensor is diagonal ( $\mathbf{A} = \text{diag}(a_1, a_2, a_3)$ ), the Hess integral (3.2) can be represented in the form [13]

$$\begin{aligned} F &= \sqrt{a_2 - a_1}\sqrt{a_3 - a_2}(M_1\sqrt{a_2 - a_1} \pm M_3\sqrt{a_3 - a_2}) - \\ &- (K_1\sqrt{a_3 - a_2} \pm K_3\sqrt{a_2 - a_1}) = 0 \end{aligned} \quad (3.9)$$

We will now present the well-known Hess cases in the equations for the dynamics of a rigid body with Hamiltonian (3.1).

*The case of a single force field:*  $U = U(\boldsymbol{\gamma}), \mathbf{W} = \mathbf{W}(\boldsymbol{\gamma})$ .

1.  $U(\boldsymbol{\gamma}), \mathbf{W}(\boldsymbol{\gamma}) = 0$  is a special Hess case of the Euler–Poisson equations (see above).
2.  $U(\boldsymbol{\gamma}) = \mu\gamma_3, \mathbf{W} = (ca'_{13}, ca'_{23}, k_3), k_3 = \text{const}$  is a special case of Sretenskii integrability [13].
3.  $U(\boldsymbol{\gamma}) = (\boldsymbol{\gamma}, \mathbf{C}\boldsymbol{\gamma}), \mathbf{C} = \text{diag}(c_1, c_2, c_3), \mathbf{W} = 0$  is a special case of the integrability of Kirchoff's equations which was pointed out for the first time by Chaplygin [14] from an analysis of the conditions for the existence of invariant relations which are linear in the momenta. The same result was later obtained in [11] using the method of splitting of the separatrices.

*The case of two force fields when  $\mathbf{W} = 0$*  [12]. Two special cases of system (3.5) were considered in [12] but the question of integrability was not discussed. The Hess integral is written in the form  $M_3 = 0$ .

$$1. \quad U = \frac{1}{2}(c_{11}^{(1)}(\alpha_1^2 + \alpha_2^2) + c_{33}^{(1)}\alpha_3^2) + \frac{1}{2}(c_{11}^{(2)}(\beta_1^2 + \beta_2^2) + c_{33}^{(2)}\beta_3^2)$$

In this case, the Hamiltonian of the reduced system (3.6) can be represented in the form

$$H = \frac{1}{2}a'_{11}\mathbf{N}^2 + \frac{1}{2}(c_{33}^{(1)} - c_{11}^{(1)})p_1^2 + \frac{1}{2}(c_{33}^{(2)} - c_{11}^{(2)})p_2^2$$

By virtue of the relation  $(\mathbf{N}, \mathbf{p}) = M_3 = 0$ , this case is isomorphic with the Neumann system, which is integrable.

$$2. U = r_3 \alpha_3 + \frac{1}{2}(c_{11}(\beta_1^2 + \beta_2^2) + c_{33}\beta_3^2)$$

The Hamiltonian of the reduced system has the form

$$H = \frac{1}{2}a_{11}'\mathbf{N}^2 = r_3 p_1 + \frac{1}{2}(c_{33} - c_{11})p_2^2$$

and corresponds to a spherical pendulum in a gravity field and in a Brun field perpendicular to it.

*Remarks.* 1. The Hess integral, like the Lagrange integral, exists in a more complex system with five degrees of freedom [15]; a body suspended on a weightless rigid rod (string) moves in a gravity field. A further three involute integrals do not suffice for the integrability of this system, even when the above-mentioned integrals exist. They are unknown, and the sole case of integrability is associated with a complete separation of the motions when the point where the body is fixed to the string coincides with the centre of mass [16].

2. Generalizations of the invariant Hess relation to the motion of chains of heavy rigid bodies connected by hinges have been studied [17] and the conditions for the existence of semiregular precessions have been given.

#### 4. THE MOTION OF A RIGID BODY ALONG A SMOOTH PLANE

The equations of motion of a rigid body along a smooth plane can also be represented in Hamiltonian form in an  $e(3)$  algebra (1.2) with Hamilton function [9]

$$H = \frac{1}{2}(\mathbf{A}(\mathbf{M} - \mathbf{K}), \mathbf{I}\mathbf{A}(\mathbf{M} - \mathbf{K})) + \frac{1}{2}m(\mathbf{a}, \mathbf{A}(\mathbf{M} - \mathbf{K})) + U(\boldsymbol{\gamma}) \tag{4.1}$$

$$\mathbf{a} = \mathbf{r} \times \boldsymbol{\gamma}, \quad \mathbf{A} = (\mathbf{I} + m\mathbf{a} \otimes \mathbf{a})^{-1}$$

where  $\mathbf{K}$  is the vector of the gyrostatic moment which is constant in the body,  $\boldsymbol{\gamma}$  is the vector of the normal to the surface,  $\mathbf{M}$  is the kinetic momentum vector which is associated with the angular velocity by the formula

$$\mathbf{M} = \mathbf{I}\boldsymbol{\omega} + m\mathbf{a}(\mathbf{a}, \boldsymbol{\omega}) \tag{4.2}$$

$\mathbf{I}$  is the constant matrix of the moments of inertia of the body with respect to the centre of mass and  $m$  is the mass of the body.

The vector  $\mathbf{r}(\boldsymbol{\gamma})$  can be found from the equation

$$\boldsymbol{\gamma} = -\text{grad} F(\mathbf{r}) / |\text{grad} F(\mathbf{r})|$$

where  $F(\mathbf{r}) = 0$  is the equation of the surface of the body.

*Theorem* [12]. Suppose a body is bounded by an axially symmetric surface, the axis of symmetry of which is perpendicular to the circular cross-section of a gyration ellipsoid of the form

$$(\mathbf{M}, \mathbf{I}^{-1} \mathbf{M}) = \text{const}$$

We shall select a system of coordinates one of whose axes ( $Ox_3$ ) is perpendicular to a circular cross-section and the other axis ( $Ox_2$ ) is directed along the central axis of inertia. Then, if the potential energy depends solely on  $\gamma_3$  and the relations

$$K_2 = 0, \quad a_{11}^{(0)} K_1 + a_{13}^{(0)} K_3 = ca_{13}^{(0)}$$

are satisfied where  $\mathbf{A}^{(0)} = \|a_{ij}^{(0)}\| = \mathbf{I}^{-1}$ , then the invariant Hess relation takes the form

$$M_3 - c = 0$$

*Remark.* In the chosen system of coordinates  $a_{11}^{(0)} = a_{22}^{(0)} = a_{13}^{(0)} \neq 0, a_{12}^{(0)} = a_{23}^{(0)} = 0$ , and the equation of the body surface and the vector  $\mathbf{a}$  have the form

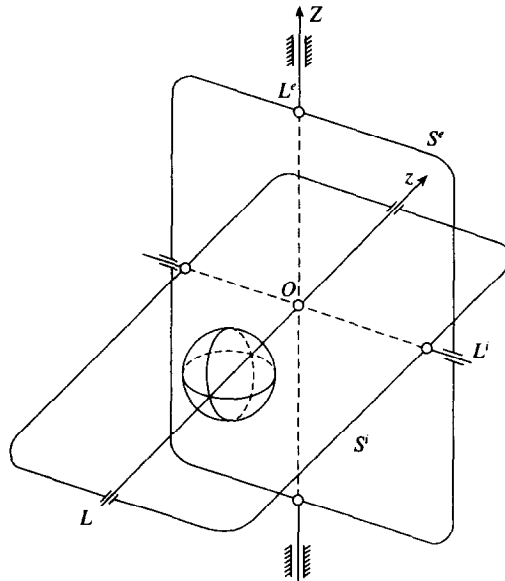


Fig. 2

$$F = F(x_1^2 + x_2^2, x_3) = 0, \quad \mathbf{a} = (-f(\gamma_3)\gamma_2, f(\gamma_3)\gamma_1, 0)$$

where the function  $f$  depends solely on  $\gamma_3$ .

For a rigid body on a smooth surface, the Hamiltonian in the Hess case differs from the Hamiltonian in the Lagrange case in that there is an additional term of the form  $(M_3 - c)f(\mathbf{M}, \boldsymbol{\gamma})$ . This term vanishes at the level of the Hess integral, at which a transition to a reduced system, which is determined by the variables (2.3), is also possible.

### 5. A GYROSCOPE IN GIMBALS

A gyroscope in gimbals is a system of several bodies coupled to one another by cylindrical hinges (Fig. 2) [18, 19].

We shall consider the case most frequently encountered in practice when the axes  $L^e$  and  $L^i$ ,  $L$  and  $L^i$  are mutually perpendicular and intersect at a single point  $O$ . We choose a fixed system of coordinates with origin at the point  $O$  and with the  $OZ$  axis directed along the axis of rotation  $L^e$ , and we associate with the body a moving system of coordinates with origin at the point  $O$  and with the  $OZ$  axis directed along the axis  $L$ . Suppose  $\alpha$ ,  $\beta$  and  $\gamma$  are the projections of the unit vectors of the fixed space onto the axes associated with the body, where the vector  $\boldsymbol{\gamma}$  corresponds to the  $OZ$  axis.

The Lagrangian for a gyroscope in a potential field can be written in the form

$$L = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) + \frac{1}{2}I^e \left( \frac{\omega_1\gamma_1 + \omega_2\gamma_2}{\tilde{\gamma}_2} \right)^2 + \frac{1}{2\tilde{\gamma}_2} \left[ I_1^i (\omega_1\gamma_2 - \omega_2\gamma_1)^2 + (\omega_1\gamma_1 + \omega_2\gamma_2)^2 \left( I_2^i + I_3^i \frac{\gamma_3^2}{\tilde{\gamma}^2} \right) \right] - U(\alpha, \beta, \gamma), \quad \tilde{\gamma}^2 = \gamma_1^2 + \gamma_2^2 \tag{5.1}$$

where  $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$  are the projections of the angular velocity onto the axes associated with the body,  $\mathbf{I}$  is the tensor of the moments of inertia of the rigid body about the point  $O$ ,  $I^e$  is the moment of inertia of the frame  $S^e$  about the axis  $L^e$ , and  $I_1^i, I_2^i, I_3^i$  are the principal moments of inertia of the internal frame.

The Hamiltonian form of system (5.1) can be obtained using the Legendre transformation

$$\mathbf{M} = \partial L / \partial \boldsymbol{\omega}, \quad H = (\mathbf{M}, \boldsymbol{\omega}) - L|_{\boldsymbol{\omega} \rightarrow \mathbf{M}}$$

Here, the Hamiltonian of the system in the general case has a cumbersome form. We will present it assuming that the body is symmetrical about the axis  $L$  ( $a_2 = a_1$ )

$$\begin{aligned}
H &= \frac{1}{2}a_3M_3^2 + \frac{1}{2}a_1k(M_1^2 + M_2^2) + \frac{1}{2}a_1^2k \left[ I_1^i(M_1\gamma_1 + M_2\gamma_2)^2 + \right. \\
&+ \left. \left( I^e + (I_3^i - I_2^i)\frac{\gamma_3^2}{\tilde{\gamma}^2} \right) (M_1\gamma_2 - M_2\gamma_1)^2 \right] + U(\alpha, \beta, \gamma) \\
k &= \left( (1 + a_1I_1^i) \left( 1 + a_1I^e + a_1(I_3^i - I_2^i)\frac{\gamma_3^2}{\tilde{\gamma}^2} \right) \right)^{-1}, \quad \tilde{\gamma}^2 = \gamma_1^2 + \gamma_2^2 \\
\mathbf{A} &= \mathbf{I}^{-1} = \text{diag}(a_1, a_2, a_3)
\end{aligned} \tag{5.2}$$

For this system, an invariant relation of the Hess type also exists provided that the axis along which an asymmetric rigid body is fixed in the internal frame  $S^i$  (Fig. 2) coincides with a perpendicular to the circular cross-section of the ellipsoid of revolution and the potential energy has the form  $U = U(\alpha_3, \beta_3, \gamma_3)$ . By choosing this axis as the  $Ox_3$  axis of the moving system of coordinates, the Hess integral can be represented in the form

$$M_3 = 0 \tag{5.3}$$

At the same time, the Hamiltonian differs from the Hamiltonian in the Lagrange case (5.2) in that there is a term of the form  $M_3f(\mathbf{M}, \alpha, \beta, \gamma)$ . In this case, a reduction of the system using the variables (2.3) is also possible.

## 6. THE HESS CASE IN CHAPLYGIN'S EQUATIONS

We will now discuss a further case of the existence of an invariant Hess relation for a system which describes the fall of a rigid body in a fluid without an initial momentum [20]. Suppose the surface bounding the body is axially symmetric and the axis of symmetry is perpendicular to a circular cross-section of the gyration ellipsoid, which is similar to the conditions in the problem on the dynamics of a body on a smooth surface. The Hamiltonian can be represented in the form

$$H = \frac{1}{2}(M_1^2 + M_2^2 + aM_3^2 + 2a_{13}M_1M_3) + \frac{1}{2}\mu t^2\gamma_3^2 \tag{6.1}$$

The invariant relation in this case also has the form (5.3). As above, the dynamics of the angle of nutation is the same as in the Lagrange case. Taking account of the invariant relation (5.3), we find

$$-\sin\theta\ddot{\theta} = -c\cos\theta/\sin^2\theta - \mu t^2\sin^2\theta\cos\theta, \quad c = (\mathbf{M}, \gamma) \tag{6.2}$$

When  $c = 0$  a well-known equation is obtained which has been studied by Chaplygin [14] and Steklov [21].

Nevertheless, the angle of natural rotation cannot be obtained in this case by a single quadrature and is determined by the system

$$\dot{\varphi} = -\frac{c\cos\theta}{\sin^2\theta} + a_{13}M_1, \quad \dot{M}_1 = a_{13}M_1M_2 + \mu t^2\gamma_2\gamma_3$$

where

$$\gamma_3 = \cos\theta, \quad \gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi$$

and the quantity  $M_2$  can be found from the relation

$$M_1^2 + M_2^2 = c^2 + \gamma_3^2/(1 - \gamma_3^2)$$

The results of a previous qualitative analysis for the plane-parallel motion of a plate [22] hold for this case.



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